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Partial Regularity and the Navier-Stokes equations

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I wish to report on recent joint work with L. Nirenberg and L. Caffarelli [1], in which we prove

Theorem: The singular set of a "suitable weak solution" of the Navier-Stokes equations in three space dimensions has "parabolic one-dimensional measure zero" in space-time.

This theorem strengthens the results V. Scheffer [3-7]. The "suitable weak solutions" we study are like Leray-Hopf weak solutions, but they satisfy a generalization of the usual energy inequality: if $\phi > 0$ is C^∞ and compactly supported in space time then

$$(1) \qquad 2\iint |\nabla u|^2\phi < \iint |u|^2(\phi_t+\Delta\phi) + \iint(|u|^2+2p)u\cdot\nabla\phi + 2\iint(u\cdot f)\phi,$$

where u is the velocity, p is pressure, and f is the external force:

$$(2) \qquad u_t + u\cdot\nabla u - \Delta u + \nabla p = f$$
$$\nabla\cdot u = 0 \; .$$

The singular set of u is

$$S = \{(x,t): u \text{ is not } L^{\infty}_{loc} \text{ in any neighborhood of } (x,t)\}.$$

To say that "S has parabolic one-dimensional measure zero" means that for any $\varepsilon > 0$ there is a finite family of parabolic cylinders

$$Q_{r_i}(x_i,t_i) = \{(y,\tau): |y-x_i| < r_i, \; |\tau-t_i| < r_i^2\}$$

satisfying $S \subseteq \bigcup_i Q_{r_i}(x_i,t_i)$ and $\Sigma r_i < \varepsilon$.

The proof of the theorem draws heavily from Scheffer's method in [4]. There are three main steps:

Step 1: There is a minimum rate at which singularities can develop. The precise statement of what we prove is somewhat technical, and I do not repeat it here. Heuristically, however, it says that if

$$R(r;x,t) = \frac{1}{\mathrm{vol}(Q_r)} \iint_{Q_r(x,t)} (|u|r)^3 \, dxdt$$

is small enough, then $|u|$ is bounded on $Q_{r/2}(x,t)$. Since we have set viscosity = 1 in (2), $R(r;x,t)$ is dimensionless; one should think of it as a <u>local Reynolds number</u>.

<u>Step 2</u>: $|\nabla u| \to \infty$ <u>as</u> $1/r^2$ <u>near a singular point</u>. Step 1 suggests that as $r \to 0$

$$|u|(x,t) > C/r, \qquad r = |x-x_0| + |t-t_0|^2$$

if (x_0,t_0) is a singular point. One expects, then, on dimensional grounds, that $|\nabla u|^2 > C/r^2$. We have proved the following estimate: <u>if</u> $\limsup_{r\to 0} r^{-1} \iint_{Q_r(x,t)} |\nabla u|^2 \, dxdt$ <u>is small enough, then (x,t) is a regular point</u>.

<u>Step 3</u>. <u>S has parabolic one-dimensional measure zero</u>. This follows easily from step 2, using a Vitali-type covering lemma.

An expository discussion of this work will appear in [2]; the mathematical details are in [1].

References

[1] Caffarelli, L.; Kohn, R.; Nirenberg, L.; "Partial regularity of suitable weak solutions of the Navier-Stokes equations," Comm. Pure Applied Math, to appear.

[2] Kohn, R., "Partial regularity and the Navier-Stokes equations," in U.S.-Japan Seminar on Nonlinear Partial Differential Equations in Applied Science, Tokyo, 1982 (to appear).

[3] Scheffer, V., "Partial regularity of solutions to the Navier-Stokes equations," Pac. J. Math. vol 66, 1976, pp. 535-552.

[4] Scheffer, V., "Hausdorff measure and the Navier-Stokes equations," Comm. Math. Phys., vol 55, 1977, pp. 97-112.

[5] Scheffer, V., "The Navier-Stokes equations in space dimension four," Comm. Math Phys., vol 61, 1978, pp. 41-68.

[6] Scheffer, V., "The Navier-Stokes equations on a bounded domain," Comm. Math. Phys., vol 73, 1980, pp. 1-42.

[7] Scheffer, V., to appear in Comm. Math. Phys.